Introduction

When considering what it means to be an expert, a variety of attributes come to mind. For example, a degree or a prestigious award in one's area of expertise seemingly puts one into the "expert" category. But what does it truly mean to be an expert? Further, why does expertise matter, and what are the implications for teachers? For this paper, I interviewed an expert and a novice over a mathematical topic to examine their proficiencies and analyze how their responses to the questions compare to literature on the characteristics of experts and novices. I also address how this information applies to classroom practices.

My novice, referred to as "S" in the transcript, is a high school senior in Kansas. The highest math course they have taken is calculus. My expert, referred to as "M" in the transcript, obtained his master's degree in mechanical engineering. He is currently pursuing a PhD in mechanical engineering, and his research is in mesh generation. This is also his 6th semester as a graduate teaching assistant for undergraduate mathematics courses.

Big Ideas

To compare the thought processes of the novice and the expert, I first needed to choose a mathematical topic. I went through various math textbooks and brainstormed with classmates to identify a "big idea" I wanted to base my interview questions on. A big idea is useful for, "connecting and organizing many facts, skills, and experiences; serving as the linchpin of understanding" (Wiggins, 2005, p. 69). Big ideas, although not simple in and of themselves, can be presented in a way that is accessible to those just beginning the topic. They can also be expanded upon over time so that a student's knowledge of the topic becomes both wider and deeper. I chose to write my interview questions over "properties of triangles". This is a big idea because these properties are foundational for many topics in mathematics. Basic properties, such as the number of sides and general shape, are taught as young as kindergarten (Common Core Standards Writing Team, 2013). This concept is then developed by introducing categories of triangles, internal angle measures, and so on. Properties of triangles are also the basis for much of calculus, which then extends into the sciences in the form of physics.

Interview Protocol

After choosing “properties of triangles” as my big idea, the next step was to write questions for my expert and novice to answer. My questions needed to address the five mathematical proficiencies and metacognition. At least one question needed to be ambiguous. Metacognition refers to a person's ability to think about their learning and understanding of a topic (Bransford, Brown, and Cocking, 2000, p. 12). The five mathematical proficiencies are:

- Conceptual Understanding
- Procedural fluency
- Strategic Competence
- Adaptive Reasoning
- Productive Disposition

Conceptual Understanding is a student’s ability to understand the underlying concepts behind a mathematical topic. Procedural Fluency is a student’s ability to accurately utilize formulas and compute answers. Strategic Competence refers to a student’s ability to use their prior knowledge in new situations, such as solving word problems. Adaptive Reasoning is a student’s ability to explain their reasoning for how they solved a problem. Productive Disposition refers to a student believing in their math skills and viewing math as something that is valuable (National Research Council, 2001, pp. 115-155).

My questions and the requirements they address are as follows:

Question 1:

"What is your grade level/highest level of education?"

This is a general information question and does not address any of the proficiencies.

Question 2:

"In general, rate your math skills from 1-10. Why?"

Question 3:

"Specific to triangles, rate your math skills from 1-10. Why?"

Both my second and third question address metacognition because the interviewee is being asked to evaluate how well they can perform mathematical tasks.

Question 4:

"What are some properties of triangles?"

This question addresses conceptual understanding because it requires interviewee to have an understanding of triangle properties. It is ambiguous because while there are several right and wrong answers, there is not just one correct answer.

Question 5:

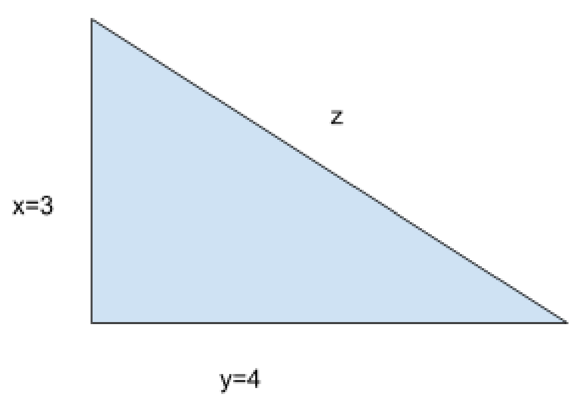
"State whether it is possible to form a triangle with the given information. Justify your reasoning."

1. ∠A=66° ; ∠B=93° ; ∠C=21°
2. ∠A=90° ; ∠B=30° ; ∠C=90°
3. ∠A=1° ; ∠B=3° ; ∠C=176°

This question addresses conceptual understanding because it requires knowledge of the property that triangle angles must add up to 180 degrees. The question addresses adaptive reasoning because the interviewee will be asked to explain their reasoning behind their answer.

Question 6:

The Pythagorean Theorem states a^2+b^2=c^2. Given this right triangle, find side length z. Show your work.



How are z and c related? Justify your reasoning. (Chalerian, Calvin, and Crabil, 1998, p. 274)

This question addresses procedural fluency because the interviewee will need to accurately utilize a formula to solve for the missing side length. It addresses conceptual understanding because the question does not state that c is the hypotenuse. Therefore, prior understanding that z and c are the hypotenuse is needed to accurately answer this question. The questions addressing adaptive reasoning because it asks the interviewee will be asked to justify how z and c are related.

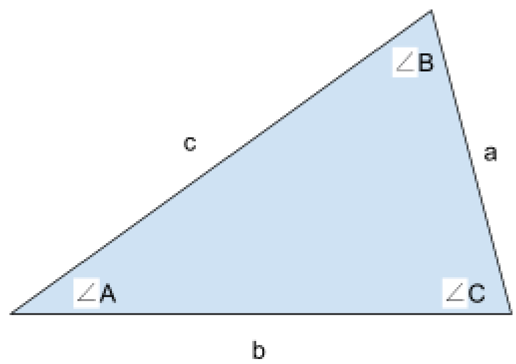
Question 7:

"Without measuring, is it possible to find missing side lengths of a non right triangle? If so, describe possible methods. If not, describe why it is not possible."

This question addresses conceptual understanding because it requires the interviewee to understand properties of non right triangles. It also addresses adaptive reasoning because the interviewee is asked to justify their answer.

Question 8:

"Law of sines states sin(∠A)/a=sin(∠B)/b=sin(∠C)/c. Given the information, find the missing angle measures and side lengths. Round answers to the nearest tenth."



$\angle A=23°$; $\angle B=51°$; $a=8$

This question addresses conceptual understanding because the interviewee needs to understand the angle properties of triangles to find the missing angle. It addresses procedural fluency because the interviewee will need to accurately utilize a formula to solve the problem.

Question 9:

"(Part 1) A ladder is leaned against a wall. The top of the ladder and the wall meet at a 45° angle, 9 ft from the ground.

a. Draw a model of this scenario.
b. Find the length of the ladder. Round to nearest hundredth.
c. Justify your answer.

(Part 2) Joe is walking and stops near the ladder. The angle of elevation from the bottom of his shoes to the top of the ladder is 30°.

a. Draw a model of this scenario.
b. Find the distance from the bottom of the ladder to Joe. Round to nearest hundredth.
c. Justify your answer.

How comfortable would you feel helping someone solve this problem? Why?”

This question addresses conceptual understanding because the interviewee needs to understand the angle properties of triangles to find the missing angles. It addresses procedural fluency because the interviewee will need to accurately use formulas to solve this problem. It addresses strategic competence because the interviewee will not be given the necessary formulas and they will have to pick out the relevant information and problem solve. It further addresses strategic competence because the interviewee will have to draw a model of the situation. It addresses adaptive reasoning because the interviewee has to explain the process they used to find the answers. Finally, it also addresses productive disposition because this question is both longer and more complicated than the others and will test the motivation to solve the problem. It also requires the interviewee to reflect on how confident they are in their ability to help someone else, further testing their productive disposition.

Analysis

Novice

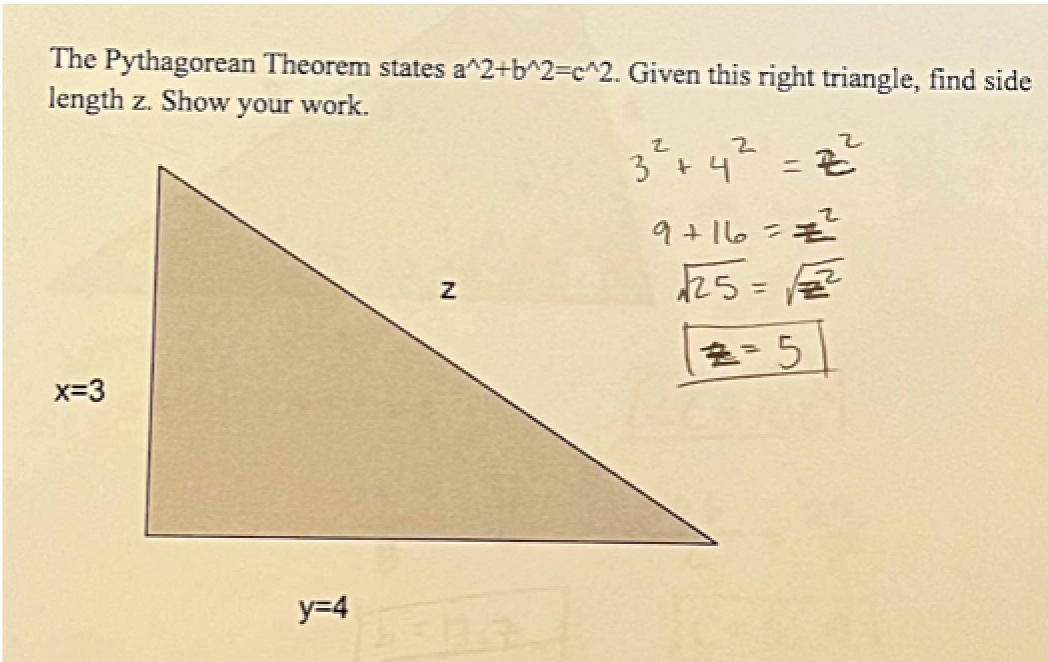
For questions 2 and 3, the novice was asked to rank their mathematical abilities and also provide a reason as to why they believed that to be their ranking. In general the novice rated both their overall math skills and skills related to triangles an 8 (Lines 21, 28). As for their general math skills, they also believed they were an 8, this time because of their ability to understand where formulas come from and apply them (Lines 21-23). They rated their math skills in regards to triangles based on their memory of trigonometric identities (Lines 27-29). This shows metacognition because of their ability to provide specific reasons as to why they gave themself those rankings.

For question 4, they were asked to state the first properties of triangles that came to mind. At first they paused and struggled to think of any, stating, “Um, maybe I’m overthinking what that means” (Line 34). However, they then went on to list the total measure of the interior angles, the amount of sides, and mentioned triangle ratios (Lines 35-37). This question was not specifically written to address the *Productive Disposition* proficiency, however, their hesitancy indicates they question themselves and their ability to answer the question adequately.

For question 5, the novice determined whether the angles provided could make a triangle based on if the angles added up to 180° (Lines 48-49). When asked why the angles had to add up to 180°, they stated, “Um, that’s the rule for triangles” (Line 53). This shows that they utilized a procedure without fully understanding why they used it. This is consistent with research on how novices tend to approach problem solving. Novices will often focus on formulas and procedures rather than the underlying concepts or principles of a subject (Bransford, Brown, Cocking, 2000, p. 37).

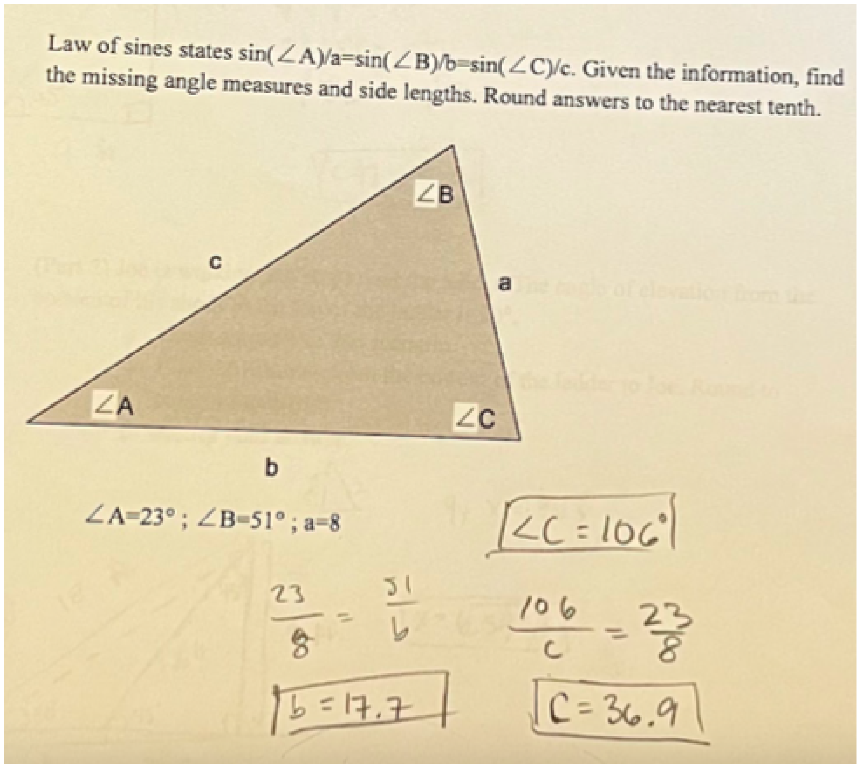
For question 6, the novice indicated that they did not like showing their work (Line 57). This behavior relates to Productive Disposition. One explanation is that they are not motivated enough to show their work. Another could be that they are not truly confident in their method and are reluctant to have to show what they did in detail.

The novice used the Pythagorean Theorem to solve for the missing side length. When asked how C in the given equation for Pythagorean Theorem and Z given in the problem related, they stated that Z and C were the same thing because they both represented the hypotenuse (Lines 67-68). When asked to justify how they knew that Z was the hypotenuse, they said the hypotenuse was opposite the 90° of a right triangle. This was a more detailed justification than that provided for the previous question. This indicates that the use of the Pythagorean Theorem was not chosen at random, but rather because they believed it to be the formula necessary to solve for the hypotenuse of a triangle.



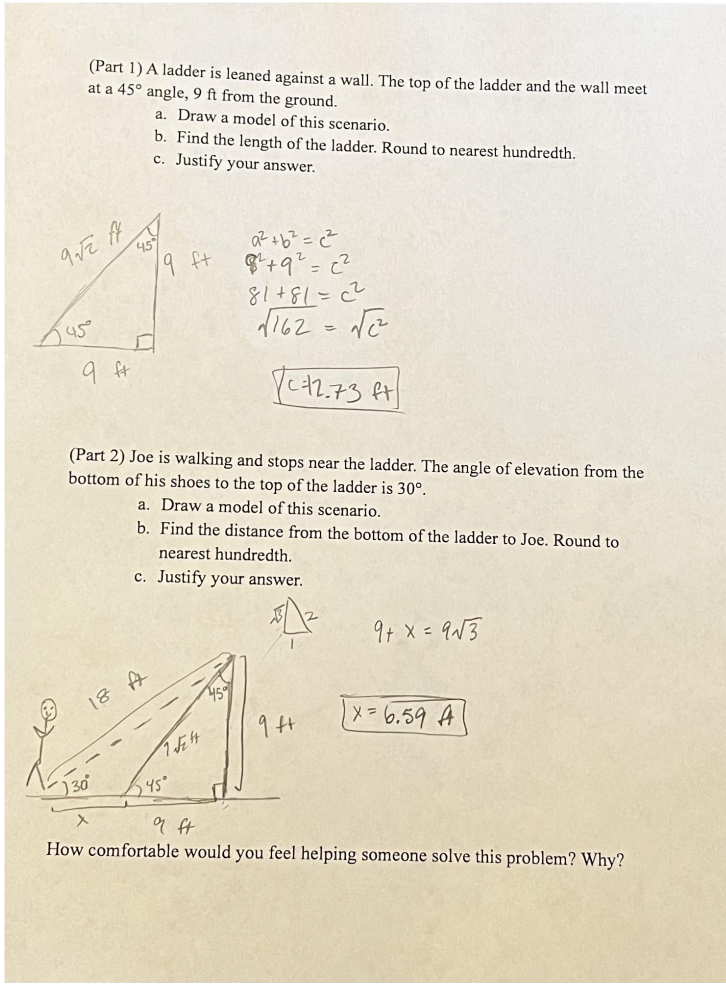
For question 7, the novice was asked if it was possible to find missing side lengths of a non-right triangle without measuring. Their response was that it could be possible if given some information, like some side lengths or angles (Lines 79-80). They elaborated that the Law of Sines or Law of Cosines could be used (Lines 85-87). Admittedly, this proved to be a confusing and unhelpful question for this analysis. This will be addressed further in the analysis of the expert responses.

While explaining their answers for question 8, they paused and said they made a mistake (Lines 135-136). I then asked them what they thought they did wrong and how their approach affected the final answer. They were able to explain their mistake in using the formula, but struggled to explain why this difference affected the answer (Lines 148-150). This shows that their understanding is based more on accurate/inaccurate use of the formula, rather than the concepts that allow the formula to solve the problem (Bransford, Brown, and Cocking, 2000, p. 38).



While solving question 9, the novice again mentioned how they disliked showing work. When asked why, they responded that, “I just know the answer” (Line 183). With this added context, it actually seems like a lack of confidence in explaining their answer is the reason behind their dislike of showing work. They lack confidence not because they got the answer wrong, but rather in their overall understanding of the concept.

When asked why they chose the method they did to solve the problem, they said that it was the first one that came to their mind (Line 251). This is consistent with literature that describes how novices approach problems. Novices tend to pick the first formula that comes to their mind and apply it, rather than considering different possibilities (Chi, 1982, p. 19).


(Part 1) A ladder is leaned against a wall. The top of the ladder and the wall meet at a 45° angle, 9 ft from the ground.

a. Draw a model of this scenario.
b. Find the length of the ladder. Round to nearest hundredth.
c. Justify your answer.

$a^2 + b^2 = c^2$

$9^2 + 9^2 = c^2$

$81 + 81 = c^2$

$\sqrt{162} = \sqrt{c^2}$

$\boxed{c = 12.73 \text{ ft}}$

(Part 2) Joe is walking and stops near the ladder. The angle of elevation from the bottom of his shoes to the top of the ladder is 30°.

a. Draw a model of this scenario.
b. Find the distance from the bottom of the ladder to Joe. Round to nearest hundredth.
c. Justify your answer.

$9 + x = 9\sqrt{3}$

$\boxed{x = 6.59 \text{ ft}}$

How comfortable would you feel helping someone solve this problem? Why?

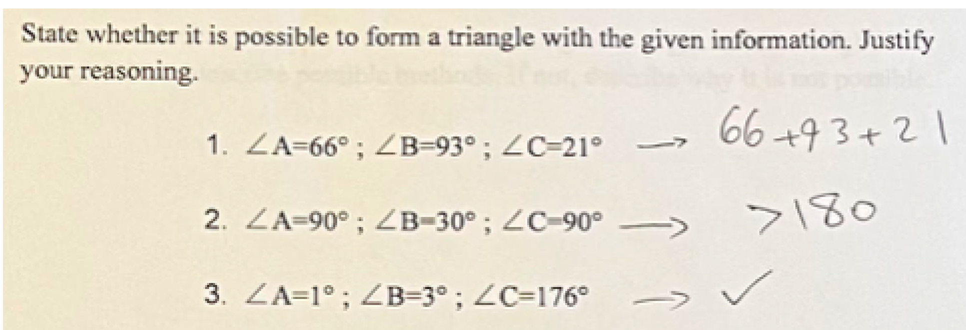
Expert

For question 2, the expert stated that his general math skills would be rated as a 10, and specified it would be a 9 if referring to advanced math (Lines 26-27). This response shows metacognition because he was able to distinguish how his skills would differ across different fields.

For question 3, he responded with a 10, however he did not explain further (Line 31). This makes it harder to judge his metacognition on this question.

For question 4, he responded immediately and noted that the first thing that came to his mind were the angle measures, shape and number of sides (Lines 36-37). He also mentioned that his research came to mind because he works with triangles in mesh generation (Lines 41-42).

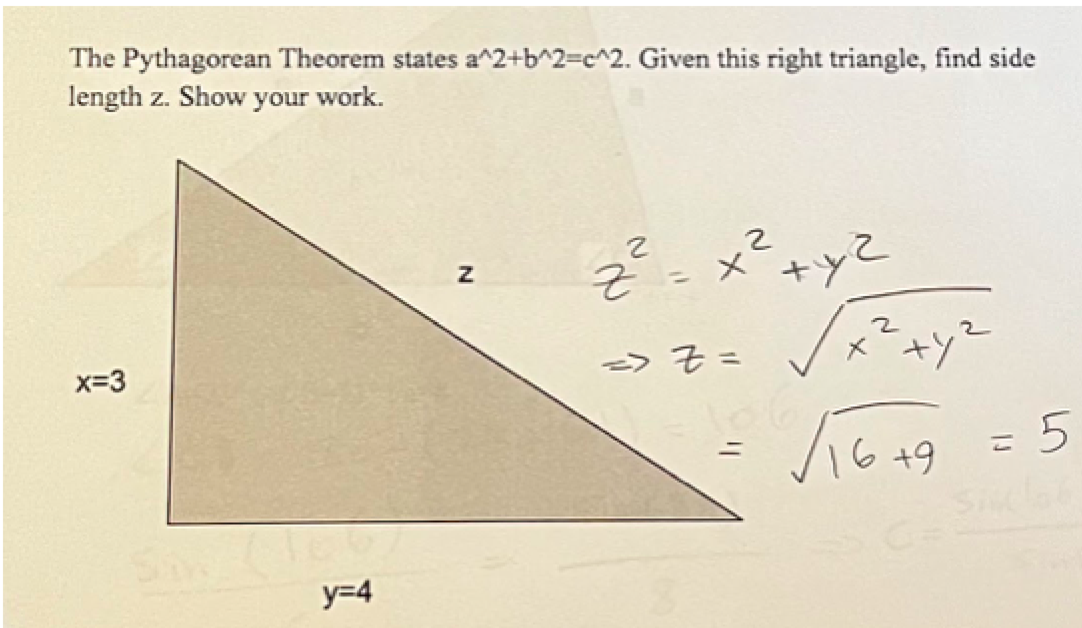
For question 5, the expert solved this by adding up the angles to see if they added up to 180°. He was able to explain why he used this method, and did so by explaining the relationship of a triangle being circumscribed in a circle (Lines 64-67). This indicates a well developed schema for properties of triangles. Schemas are, "Mental systems or categories of perception or experience" (Woolfolk, 2007, p. 28). Experts will tend to have well-developed schemas for a topic. This means that they will clearly understand the connections between different ideas and how they come together to form a concept (Bransford, Brown, Cocking, 2000, p.33).


State whether it is possible to form a triangle with the given information. Justify your reasoning.

1. $\angle A=66°$; $\angle B=93°$; $\angle C=21°$ $\rightarrow$ 66+93+21
2. $\angle A=90°$; $\angle B=30°$; $\angle C=90°$ $\rightarrow$ >180
3. $\angle A=1°$; $\angle B=3°$; $\angle C=176°$ $\rightarrow$ ✓

For question 6, the expert was able to elaborate on information that was not immediately present in the question. He used the Pythagorean Theorem to solve this problem. This included taking a square root to find the side lengths. He brought up that in this case, one would only want to take the positive square root, as the problem is related to finding a length (Lines 75-77). Mentioning this shows that he understands mathematical ideas that are not readily apparent in the question, but are necessary to finding the answer. This highlights his conceptual understanding of both Pythagorean Theorem and square roots.

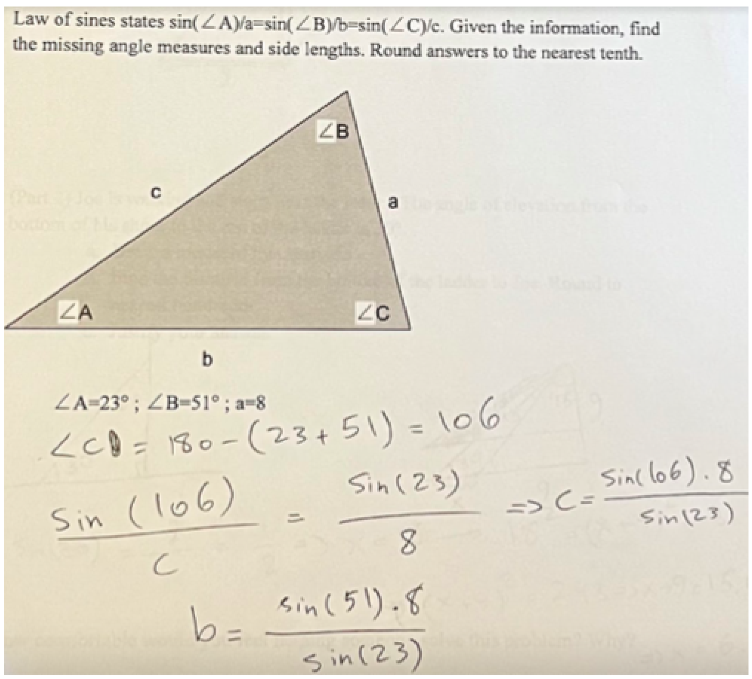
When asked why he used Pythagorean Theorem to solve the problem, he admitted that he did not know, it was just the way he was taught (Lines 95-96). This is an interesting comment, because it actually strays from what literature about experts states. Generally, experts tend to consider all of the variables in the problem to decide what mathematical principles apply. In this case it does not appear that the expert did this, but rather went with the first approach that came to him (Bransford, Brown, Cocking, 2000, p.38).



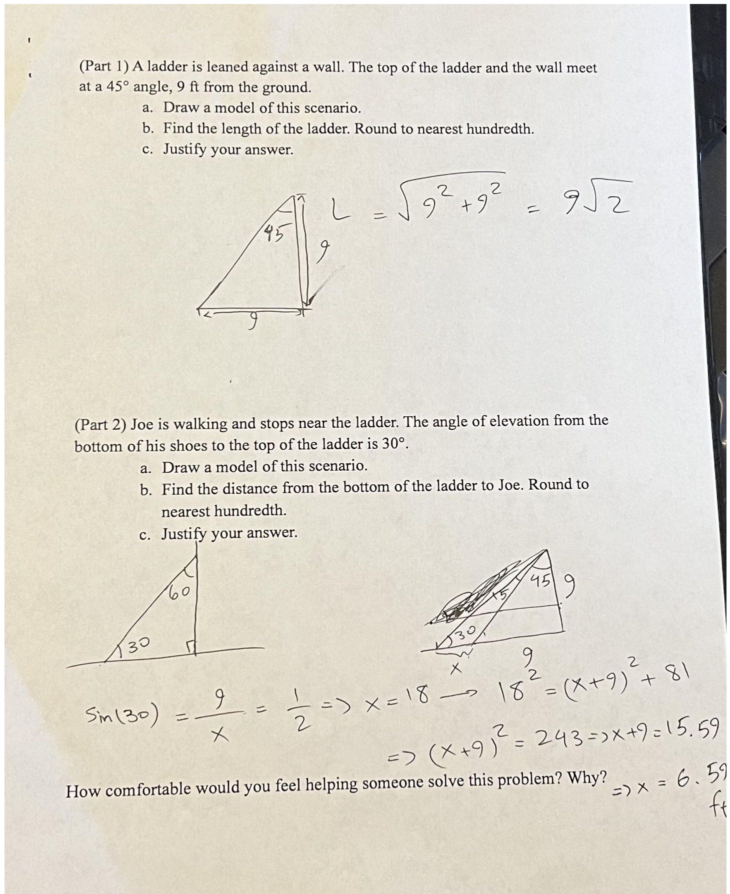
Regarding question 7, the expert at first answered that it would not be possible to solve for the missing side lengths (Line 104-105). It was only after I prompted him with too much information that he said it could be possible (Lines 107-110). This question did not end up being helpful because it was not very clear and also did not prompt good discussion. The effect was that it essentially became a yes or no question that was difficult to discuss further without hints.

For question 8, the expert did not put his answers into the calculator to solve; he presented the ratio for what the side lengths would be. When asked why he did this rather than solving, he said he believed the answer was clear (Lines 145, 150-151). This shows a greater focus on the concept than the answer.

When asked why he used Law of Sines to solve the problem, he mentioned that he considered using Law of Cosines. However, he decided there was not enough information to use the Law of Cosines (Lines 164-167). As previously, experts tend to approach problems by variables first and assess different approaches to solving problems. Therefore this response was more consistent with research on expert thinking.



For question 9, the expert at first did not realize the ladder in parts one and two were the same. I did confirm this, which was a hint. When completing part two, the expert did not know the term “angle of elevation” because english is not his first language and this was not the term he was taught with (Lines 219, 224-225). However, he was still able to use the context of the question, the diagram he drew, and his background knowledge to understand what the term was referring to. This is a clear example of the expert using transfer. Transfer is the ability to use previous knowledge in new situations. Research on this topic suggests, “Experts, regardless of the field, always draw on a richly structured information base; they are not just "good thinkers" or "smart people." The ability to plan a task, to notice patterns, to generate reasonable arguments and explanations, and to draw analogies to other problems are all more closely intertwined with factual knowledge than was once believed" (Bransford, Brown, Cocking, 2000, p. 16).



(Part 1) A ladder is leaned against a wall. The top of the ladder and the wall meet at a 45° angle, 9 ft from the ground.

a. Draw a model of this scenario.
b. Find the length of the ladder. Round to nearest hundredth.
c. Justify your answer.

$L = \sqrt{9^2+9^2} = 9\sqrt{2}$

(Part 2) Joe is walking and stops near the ladder. The angle of elevation from the bottom of his shoes to the top of the ladder is 30°.

a. Draw a model of this scenario.
b. Find the distance from the bottom of the ladder to Joe. Round to nearest hundredth.
c. Justify your answer.

$\sin(30) = \frac{9}{x} = \frac{1}{2} \Rightarrow x = 18 \rightarrow 18^2 = (x+9)^2 + 81$

$\Rightarrow (x+9)^2 = 243 \Rightarrow x+9 = 15.59$

$\Rightarrow x = 6.59$ ft

How comfortable would you feel helping someone solve this problem? Why?

Similarities and Differences

One similarity between my novice and my expert was their metacognitive approach to deciding what score they would give their mathematical abilities. They were both able to give reasons as to why they ranked themselves how they did. Another similarity is that both the novice and expert stated they did a procedure without exactly knowing why they chose it. More care should have been taken on my part to address this further with other interview questions. As a result, it is hard to know if this was the overall approach for either of them. However, at one point the expert was also able to point to different approaches he considered before deciding on one approach. This points to some of the differences between the expert and the novice.

One major difference between the novice and the expert was their productive disposition. Several times the novice indicated that they were not confident in their answers. This came in the form of not being sure they provided enough information for their model or by repeatedly stating they did not want to show their work (Lines 57, 183). In contrast, the expert never expressed doubt in their answers, even when directly asked. For example, the expert was asked why he believed he did not have to simplify his answer. His reasoning was that since he provided the ratio, the rest was just “the work of calculator” (Line 145). This shows that he was confident in his answer as he did not change it and he defended its correctness.

Another difference is that the expert overall showed more evidence of transfer in his responses. This is largely because of his ability to understand the meaning of an unknown term. This does not mean that the novice did not also use transfer, but it was not readily apparent through the interview. This indicates a need for better follow up questions on my part.

The final main difference is that the expert had a more developed schema for the information he knew. For the majority of the questions, he was able to provide a clear and detailed response for this method in solving the problems. He also had more ability to include information that seemed unnecessary to solve the problem, but formed the base concepts for problem solving.

Applications

It is important to analyze experts and novices because the goal of teaching is to move students towards expert thinking. To achieve this goal, there are three main teaching strategies I would use: focusing on conceptual understanding, facilitating metacognition, and engaging in culturally responsive teaching.

Conceptual Understanding

Many students learn definitions, formulas, and other procedures before they are familiar with the contexts in which they are used. This often leads to a lack of conceptual understanding and even misconceptions about the content. Further, if mathematical facts are presented in a disjointed way without clear connections, students' schemas for a topic will not be well developed. Research literature on learning states, "To develop competence in an area of inquiry, students must have opportunities to learn with understanding. Deep understanding of subject matter transforms factual information into usable knowledge" (Bransford, Brown, and Cocking, 2000, p. 16).

Lesson structure contributes greatly to facilitating conceptual understanding. Inquiry based learning is particularly effective for achieving conceptual understanding. Inquiry based learning refers to allowing students to interact with concepts before being introduced to terms or formulas. One style of lesson planning, referred to as the "5es" facilitates inquiry based learning. The 5es are as follows:

- Engage: To begin the lesson, students are given an activity that grabs their attention and introduces them to the topic

- Explore: This section allows students to dive deeper into the topic and discover concepts on their own. This could include giving students a problem that requires them to arrive at a formula on their own, rather than giving it to them at the beginning of the lesson.
- Explain: Students are then asked to explain what they found out from the explore section. This also gives the teacher an opportunity to gauge the current understanding.
- Elaborate: This section allows students to continue practicing the concepts or do extension activities.
- Evaluate: Students will be given some type of assessment to evaluate what they have learned.

Reports on the effectiveness of the 5e model state, “students whose teachers taught with medium or high levels of fidelity to the BSCS 5E Instructional Model experienced learning gains that were nearly double that of students whose teachers did not use the model or used it with low levels of fidelity” (Bybee, 2009, p. 12).

Metacognition

Metacognition, or the ability to think about your own understanding of a topic, is also key to achieving expert-level thinking. Metacognitive activities lead to better understanding of mathematical topics and ability to transfer existing knowledge to new situations (Bransford, Brown, and Cocking, 2000, p. 19). One metacognitive activity is creating concept maps. Concept maps involve connecting mathematical ideas through writing. Concepts are written and circled. Then lines are drawn between them with a verb that shows the connections between the ideas

(Baroody, 2000, p. 605). Concept maps are helpful because it provides students an opportunity to address their own understanding of a topic. If they struggle to make connections, this could indicate that their understanding is not as strong as they may have believed (Baroody, 2000, p. 607). It also gives the teacher the ability to see what their students do not understand and address misconceptions the students have. Concept maps can also be worked on throughout a unit or even an entire semester. Coming back to them as more lessons are taught can be helpful to show how student learning is progressing (Baroody, 2000, p. 608).

Culturally Responsive Teaching

Culturally responsive teaching is in part defined as, "An educator's ability to recognize students' cultural displays of learning and meaning making and respond positively and constructively with teaching moves that use cultural knowledge as a scaffold to connect what the student knows to new concepts and content in order to promote effective information processing" (Hammond, 2014, p. 15). Research shows that students are more likely to understand concepts and be able to transfer this information if they are connected to previous knowledge (Brandford, Brown, and Cocking , 2000, p. 68). Understanding deep culture and cultural archetypes are important to be engaged in culturally responsive teaching. Deep culture refers to a person's understanding of the world on a fundamental level, such as their conception of good versus evil (Hammond, 2014, p. 23).. Deep culture informs cultural archetypes, or core values that are shared among different cultures (Hammond, 2014, p. 25). An example of two cultural archetypes are cultures which have written traditions, and those which have oral traditions. These will inform how students process information (Hammond, 2014, p. 28).

Hammond continues, “All the while, the educator understands the importance of being in a relationship and having a social-emotional connection to the student in order to create a safe space for learning” (Hammond, 2014, p. 15). Students are unable to learn in an environment where they do not feel safe or welcome. This is known as “amygdala hijack”. The amygdala is the part of your brain that processes fear. When fear is detected, “all other cognitive functions such as learning, problem solving, or creative thinking stop” (Hammond, 2014, p. 40). Therefore it is imperative to create a classroom culture that is safe, welcoming, and inclusive.

[Copies of novice and expert interview transcripts appear on following pages.]

Novice Interview Transcript

For the duration of the transcript the interviewer will be referred to as 'K' and the novice will be referred to as 'S'.

[The interviewer explained the purpose of the interview. The novice was told they can solve problems however they see fit unless otherwise stated by the question. They were also allowed to use a calculator. However, the interviewer was instructed to not help the novice to solve the problem.]

K: So what is your grade level?

S: I'm a senior so, 12.

K: And what is like the highest level of math that you've taken?

S: Calculus.

K: Calculus? Um ,in general rate your math skills from 1 to 10 and then why.

S: I would give it an 8. I feel like I'm pretty good at like, memorizing formulas, but also understanding where they come from. And then creatively applying them. I'm also very good at saying things like three times four is seven. So that's why I would give an eight

K: Yeah. But specifically to triangles, rate your math skills from 1 to 10 and why.

S: That's very... Triangles. [Pause] I mean, I would assume triangles means trig. And if it does mean trig, then I would say like, around 8. There's some trig identities I definitely don't remember that well, but I remember like the basic identities.

K: Um, so this question is, what are some properties of triangles, like just some, kind of the first few that come to your mind?

S: Properties of triangles? Um, maybe I'm overthinking what that means. Okay. Properties of triangles. So the interior angles always add up to 180. There's three sides. There's like, if the angles are the same on the inside, then the ratios for different like sizes of triangles would be the same.

K: Okay, and then so number five is the actual like, kind of like math problems. So number five is, state whether it is possible to form a triangle with the given information and then justify your reasoning.

S: [Pause as S works out the problem on the paper.] I believe one should work as a triangle. Two definitely not. Three, yes.

K: Why did you decide that one and three work and then two doesn't?

S: Because those angles add up to 180 for one and three. And for two, it's like 210 or something like that.

K: Where are you getting the 180? Why does it have to add up to 180?

S: Um, that's the rule for triangles. [Laughing]

K: Okay. So number six is the Pythagorean Theorem states A squared plus B squared equals C squared. Given this right triangle, find side length z and show your work.

S: Oh I have to show my work? That's annoying. [Pause as S works out the problem on the paper.] Okay I showed my work.

K: Okay.

S: I guess this should be Z rather than C. [Their answer is z=5. Refer to pictures.]

K: And then how are Z and C related?

S: I mean, in this case, they're the same value because C represents the hypotenuse of the triangle. And z is the hypotenuse of this triangle.

K: Okay. And then how do you know that C is the hypotenuse, or Z is the hypotenuse?

S: Um, because it's a right triangle, and that's the longest side and it's opposite to the 90 degree angle.

K: Okay, um. And then number seven, so without measuring, is it possible to find side lengths of a non right triangle? If so, describe possible methods and if not describe why it is not possible.

S: Um, it's possible as long as you have like some angles, I believe. And you would need at least one side length. Just

K: What would the possible method be to using that information to solve for the [side lengths].

S: Law of sines and law of cosines, which I don't remember that well. But I know Law of sines is sine of angle A over side a is equal to sine of angle B over side b and so on. Sine of angle C over side c is also equal to those.

K: Okay. And so then number eight. [Laughing] So, Law of Sines states, and I won’t repeat it since you just. But given information, find the missing angle measures and side lengths and round your answers to the nearest 10th.

S: [Pause as S works out the problem on the paper.] I’m not used to this calculator.

K: If you have one that you're more comfortable with, you can use that one as well.

S: Oh I can? Is it okay if it's a graphing calculator?

K: Yes.

S: Okay. Because I'm not used to that calculator.

K: That’s totally, I get that.

S: This has been my precious calculator since freshman year.

K: I just wanted to make sure like, you had one if you didn't.

S: This will be a little easier. [Pause as S works out the problem on the paper.] Okay.

K: What did you get?

S: I’m just making sure I actually put all of the information, hold on.

K: Okay.

S: Yeah, okay, so I got the length of side b is about 17.7. Um, angle C, the measure of angle C is 106 degrees. And then the length of side c is 36.9.
K: Okay, how did you get angle C?

S: Well, the angles in a triangle add up to 180. So I did 180 Minus the measure of angle A and measure of angle B.

K: And then how did you get Side a and Side c?

S: Oh, well side a was given.

K: Oh sorry, side b, yes.

S: So I used sine of angle A over a is equal to... Oh, whoa, hold on a minute. I did a lot of things wrong here. [Laughing]

K: What do you think you did wrong?

S: I completely forgot about the sine part. I just did the [pointing at their paper and laughing]

K: What's the difference between...so what exactly did you do?

S: I just did the measure of the angle over the side instead of the ratio.

K: Why is it important to take the ratio?

S: Um, that ratio is like... it's, how do I explain it? It's like hmm. It's like a constant, like. Like they're supposed to be equal with that ratio. Not necessarily the ratio of that angle on this side.

K: Okay.

S: [Referring to angle C] However, that angle is right. But the side lengths I just completely forgot about the sine.

K: Okay. Um, we can still move on.

S: Okay.

K: Yeah, to part, to part nine. Or sorry, question nine. So it has two parts that are kind of like related to one another. Um, so part one is a ladder is leaned against a wall, the top of the ladder and the wall meet at a 45 degree angle, nine feet from the ground. So A is draw a model of a scenario and then B is find the length of the ladder. Round to the nearest 100th and justify your answer.

[S's math teacher, E, comes over.]

E: No pressure, just kidding.

S: [Laughing] So much geometry.

E: [Laughing] You're gonna do fine!

S: I completely forgot about the sines over there and law of [emphatically] sines.

E: You're good!

S: That was embarrassing! Oh justify my... [pause] I don't like showing my work.

K: Why don't you like showing your work?

S: This is a special triangle and I just know the answer.

K: [Laughing]

S: [Pause as S works out the problem on the paper.] Okay.

K: What'd you get?

S: 12.73 feet.

K: Okay. And can you explain kind of how you got to that answer?

S: Um, well, the Pythagoras theorem since this is a right triangle basically. Um, and because the angle where the ladder and the wall meet is 45 degrees, then the angle between the ladder and the ground also has to be 45 degrees. Which means that both of those lengths, the distance from the end of the ladder to the wall, like on the ground, and then from the ground to the top of the ladder, those both have to be nine feet.

K: Why?

S: Um, I mean basically the law of sines. Um, and also, for like, why those side lengths would be the same. And they both have to be 45 because there's already to the 90 degree angle at the base with the wall and the ground. And then we're given the 45 degrees. So 180 minus 90 minus 45 is 45.

K: And then how did you find the length of the ladder specifically?

S: Well, A squared plus B squared equals C squared. Yeah.

K: So then part two, so Joe is walking and stops near the ladder. The angle of elevation from the bottom of his shoes to the top of the ladder is 30 degrees. So again, draw a model of the scenario.

S: Okay.

K: Find the distance from the bottom of the ladder to Joe and round to the nearest 100th and justify an answer.

S: Okay. I get to draw a stick person.

K: [Laughs]

S: [Pause as S works out the problem on the paper.] I don’t know how much detail is wanted in this picture. But, I’m labeling all of the parts of the triangle as well. Okay, find the distance from...[S continues to work on the problem]. Okay, I hope this is enough justification. It might not be.

K: What’d you get for your answer?

S: Uh, 6.59 feet.

K: And how did you get that answer? Like, what like, method did you use to solve?

S: Well, if this is a 30 degree angle, then this is a special, it's like one of the special triangles. It’s a 30, 60, 90. And the ratio for like, the side lengths for those would be the shortest side is one. A hypotenuse is two and then the other side is the square root of three. And so then, if we're looking at that compared to this model that we have drawn. The wall must represent the shortest length of the special triangle because it's opposite 30 degree angle, which is the smallest angle. And then the line that's like that angle of elevation I guess is opposite from the 90 degree angle. So that has to be the hypotenuse. Which means that the distance from the bottom of Joe's foot to the wall has to be, um, the other side, that’s...you know. Oh, sorry. So that's, that's the side that would be like the square root of three so that means, like. And we can see that everything is multiplied by like nine, like it's nine times bigger. So instead of a one, instead of a length of one, we have a length of nine. So then this side must be nine square root of three. So nine plus x is equal to nine square roots of 3.

K: Why did you decide to use that method to solve instead of any other possible methods?

S: It's the first one that came to my mind.

K: Okay! Um, I think that's, I have one more question. So how comfortable would you feel helping someone solve this problem and why?

S: I mean, it would depend on how much they know already about trig and things like that. Because if they know special triangles, then I would feel fairly comfortable. I, I feel like I'm fairly good at explaining things. And I think I understand it well enough [Laughing].

K: Yeah.

S: However, if they don't know special triangles, I would not feel very comfortable with it. I just honestly don't remember that much from that long ago.

K: Yeah.

S: Yeah.

K: Okay, and that's it. Thank you so much!

Expert Interview Transcript

For the duration of the transcript the interviewer will be referred to as 'K' and the expert will be referred to as 'M'.

[The interviewer explained the purpose of the interview. The expert was told they can solve problems however they see fit unless otherwise stated by the question. They were also allowed to use a calculator. However, the interviewer was instructed to not help the expert to solve the problem.]

K: Okay, so my first question is, what is your highest level of education?

M: Highest level acquired?

K: Just like the highest. Yeah, that you've gotten, or like what you're pursuing right now.

M: A PhD, right now. Have gotten a master's in, back in 2017.

K: What did you get your master's in?

M: Mechanical engineering

K: And so in general, rate your math skills from one to ten. So one would be like the lowest and ten would be like the highest.

M: Um, if you are talking about advanced math, that is solving PDs, numerical analysis, I would say nine everything else I would say a ten.

K: Ten? Um so specific to triangles, rate your math skills from one to ten.

M: I would say a ten.

K: A ten? So now we're gonna go into kind of like, some of the more actual questions. So what are some properties of triangles, just like the first few that kind of come to your mind?

M: The angles and how they add up like, what's the total angle. Second thing would be the number of sides, the shape.

K: Okay. Anything else that kind of comes to your mind when you think of triangles?

M: Triangles? Well, my research is on mesh generation and we have a lot of triangular meshes.

K: Okay.

M: So that's why like the angles come into my mind first thing. Um, they're beautiful shapes, yeah.

K: Okay. So now I have like the actual problems for you to solve. So this one, for each one state whether it is possible to form a triangle with the given information and justify your reasoning.

M: Okay, so what I would do is just, I add up everything and see if the total adds up to 180. Um, so 66, 93, 21 that will give me... Uh, 66, 93 that is 9. 159, 160, 180. So the first one gives me triangle.

K: Okay.

M: Uh, second one doesn't because it's more than 180. The summation of the, all the angles. And the last one also gives me a triangle, but a very bad triangle with very, very sharp angles.

K: So why were you looking for them to add up to 180?

M: Um, why would I look at that? So triangles, if you can put a circle around it. I forgot the word for it in English. But they [triangles] don't cover like the whole, the whole circle. They cover half of it. So that's why the internal angles, the addition of internal angles, adds up to 180. It doesn't go all the way to 360, which is a full circle. And yeah, that's the reason.

K: Okay, so on to the next one. So the Pythagorean theorem states that a squared plus b squared equals C squared. So given this right triangle, find side length Z and show your work.

M: Okay. So, Pythagorean theorem. Uh, in the right angle triangle. You go the side length of the right angle, square, add them up. So z squared is equal to x squared plus y squared. Z squared is x squared plus y squared. Therefore z is equal if you take the square root of everything. x squared plus y square. and since this is a length, we don’t go with the positive/negative square root. And based on the given data here, that would be a square root of 16 plus nine, which is five.

K: Okay. So how are Z and C related?

M: Z and C are basically the same thing that's just a placeholder to show how Pythagorean theorem works.

K: So how do you like know for sure that z and c are the same thing?

M: Because it's, they used C in Pythagoras theorem. So I know it is about the right angle triangle finding the hypotenuse of the triangle. And on the image, they have z as the hypotenuse. So both of them refer to the same side in the same triangle.

K: Why did you choose to use the Pythagorean theorem to solve this problem?

M: Why did I choose to use?

M: Well [pause] that is a very good question. That goes very deep. This is just how they taught me, I guess. Yeah, never really thought about that. But I mean, it works.

K: Yeah. Okay. So number seven is on the next page. Without measuring, is it possible to find missing side lengths of a non right triangle? If so, describe the possible methods and if not described, why it is not possible.

M: So without measure, is it possible to find the missing side length? Non right angle triangle. To find missing side length. So it's not a right angle triangle? I would say no, it's not possible.

K: If you are given some of the information do you think it would be possible? Like if you're given some of the angles or some of the side lengths?

M: I would say it could be yes, yeah.

K: But without measuring, and other information, would it be possible?

M: No I don’t think it would be.

K: Okay so number eight says Law of Sines states the sine of angle A over a equals sine of angle B over b equals sine of angle C over c. Given the information, find the missing angle measures and side lengths and round your answers to the nearest tenth.

M: This is where I need the calculator [laughing].

M: So sine 23, 51. That is 74, 74. 180 What is that going to be? 106. So sine of 106 over c is equal to. I have a. Sine of 23 over 8. Do I need to get to the final answer or I can just say this is equal to that?

K: Yeah.

M: Yes to which one?

K: Sorry [laughing]. It just says find the missing angle measures and side lengths.

M: So c is equal to, side c is equal to sine of 106 times 8 over sine of 23. Side B is equal to, it's going to be sine of B over b. Equal to sine of 23 over eight. So this is going to be, side b is going to be sine of 51 times eight over sine of 23.What else do we need to? And the other angle which is angle C is equal to 180 minus angle A plus angle B. Which is 106. Do I have to go through sine of 51 multiplied by eight over sine of 23?

K: Oh okay, are you saying like do you need to put it into the calculator? Um, it's up to you if you want to finish it or not.

M: I think it’s good enough.

K: Okay!

M: [Laughing] It’s just the work of calculator!

K: Yeah. Why do you think that that answer is like sufficient compared to like putting into the calculator?

M: Um, I think everything's clear at this point. It's just, it's just basic mathematical calculation. Which is just multiplication and division. And then rounding to the nearest 10th would be I will look at the hundredth. What would when I have at the place of 100 greater than five, I add one to the 10th. Less than five I leave the 10th as it Is.

K: What was your thought process for like solving this problem? Like what did? Like what did you immediately think like was the formula you should use, anything like that to solve?

M: So first thing I did, I looked at the given data. Second thing I saw that I have two angles. I know the addition of all angles in a triangle. So I went for the missing angle. And the rest was just using the sine formula to find the other two missing side lengths.

K: Do you consider using any other form formula?

M: I was thinking when you were reading it I was thinking about using the cosine formula, which is a squared plus b squared minus two a b cosine if I'm not mistaken. So cosine of the angle between. but then that wouldn't be possible because I need two side lengths for that. And I only have one.

K: Why do you need two side lengths to [solve]?

M: Because of the a and b in the formula. But I want to use the cosine.

K: So the last question is like two parts. So part one, it says a ladder is leaned against a wall, the top of the ladder and the wall meet at a 45 degree angle, nine feet from the ground. So A) is draw model of scenario, B) is find the length of the ladder and round to the nearest 100th. And then C), Justify your answer. So kinda just explain.

M: So this is gonna be a right angled triangle. 90 from the ground, that gives me one of the side length of the right angle. The other one I can find it since the degree that the ladder and the wall make is 45. Tangent of 45 in the right angled triangle is equal to the opposite over adjacent side. So tangent of 45 is one. So that gives me the other side length of the right angle. Nine as well. And then I use Pythagorean to find the length of the ladder. So nine squared plus nine squared, square root of that is equal to the length of the ladder. Which is, which is nine square root of two.

K: Okay.

M: And how do I justify my answer? Well I justified based on the model, the diagram that I have.

K: How so?

M: And because of the diagram, because of the problem description. This would be a clear representation of the problem. And when I look at that, I see that it forms a right angled triangle. Assuming the wall is a straight wall, it's not at any angle.

K: Can you go into more about how you use like tangent to solve for like the various side lengths and why you decided to use tangent?

M: Yes. So in the right angled triangle, one of the nicest things about it is that you can use side lengths to do. To find the uh [snapping], to find the sine and cosine of the other angles other than the 90, which is given. And so by using them, well you know sine is opposite to the hypotenuse. And, well, why did I use tangent? [Laughing] Sorry, I got off track.

K: No, you're fine

M: So I use tangent because this 45 degree, this angle. I have one of its side length, right? Which is the adjacent one. I want to know what is the other, uh other side length of the right angle, of the right angle. Because I need both of them to find the hypotenuse, which is the length of the ladder. And when I look at this, I see that tangent formula would work very very nicely because I don't have to deal with any unknowns. Other than just one, which is the side of the opposite side. The length of the opposite side.

K: Okay, so part two is Joe is walking and stops near the ladder. The angle of elevation from the bottom of his shoes to the top of the ladder is 30 degrees. So again, draw a model of the scenario and find the distance from the bottom of the ladder to Joe, and then round to the nearest 100. And again just justify your answer.

M: Stops near the ladder. Angle of elevation, that is angle from horizon? Angle of elevation? You cannot answer [laughing].

K: [Also laughing] I can’t answer.

M: So angle of elevation, coming from a non English speaking country. So I'm going to assume that is measured from horizon. From the bottom of his shoes to the top of the ladder. Yes, so that would be 30. Again, considering a straight wall, uh that forms my right angle triangle. I get a 60 here, a 90 there. Find the distance from the bottom of the ladder. [Pause] Bottom of the ladder to Joe? I thought Joe is stopping near the ladder? Is this the same ladder as part 1?

K: Yeah, sorry. Yeah.

M: Okay [Laughing]

K: I should have clarified.

M: I was like, that's a lot of unknown. And also doesn't make sense. It's just gonna be the same answer. So the same ladder that makes the 45. So Joe stops here, the bottom of his shoes to the top is going to be 30. So I don't know what is this? This is gonna be unknown because [unintelligible, talking to himself while working]. Oh, why am I doing this? [Scratches out work]. So this is 30. This is 9. This is 9. So you have 45. You have 15 here. So 30 is going to be. So here I'm going to use sine of 30. Which in the right angle triangle is the opposite to hypotenuse. So I go nine over hypotenuse. I don't know what that is, is equal to sine of 30, which is one half. So that gives me hypotenuse of 18. Then I can use that because this added length is unknown. So I have, I have the hypotenuse. I have one of the side length of the right angle. I'm missing the other one. So I can use Pythagorean to find the other one. Pythagorean says hypotenuse is squared is equal to the addition of the right angle side lengths. So x plus nine squared plus 81. So x plus nine squared is going to be 18. Squared. Minus 81. So 324 minus 81, 243. So x plus nine is going to be equal to square root of 243. 243. Which is, round to the nearest hundredth. So this is going to be 15.59. Then x is going to be equal to 15 Minus nine, what is that? That is 6.59. So Joe is 6.59... Are these meters? Feet. Feet away from the ladder.

K: Okay. So going through how did you.. I mean I heard you talk through it. But like why did you decide to use like pythagorean theorem, or like sines, or anything like that?

M: So everything is all, is almost the same thing as the previous one. So everything I said about the previous one applies to this one as well. Except that the only difference here is to figuring out right diagram for this problem. Because as you saw earlier, I was... if you don't go with the right diagram, then you don't have a clear idea of what's going on and then you end up with the wrong answer. So I think that's the, I think that's the hardest part honestly, in all of math problems for the teaching or solving. Trying to like imagine, like visualize it. The right way will go a long way.

K: So specifically when you were talking about like, like when you said sine of 30? How did you know that that was like the accurate value for sine of 30? Or when you did it like here [pointing to paper].

M: Why is sine of 30 is one half?

K: Yes.

M: Okay, so um. So if you go to unit circle. And in unit circle, you have a circle that has a radius of one, that's why it's called unit circle. And if you measure the sine of 30, the only thing you can do is to form a right angle triangle. And because of the property of right angle triangle, which is opposite to hypotenuse is equal to sine value, you know the hypotenuse is one. And you look at the, where, where that sine, where that 30, the angle 30 crosses the circle. The circumference of the unit circle. And if you look at the axis that is representing sine value, you see that the sine value is half of the hypotenuse. So sine of 30 is going to be one half. and then you go back to this problem, to the problem and then see that you have a right angle triangle. So you go opposite to hypotenuse is one half and you have the opposite to the 30, to the angle 30. So the only thing missing is this new hypotenuse. And then you can use that relationship to find a hypotenuse and the rest is history.

K: Okay, so the last questions are how comfortable would you feel helping someone with this problem and why?

M: So I would find, I would find both of them actually comfortable, helping someone with. But I would find the second one a little more challenging in terms of coming up with the right diagram. Because the first one is a lot more clear than the second one. And that is one of the problems in math. Like when the problem, when the question gets wordy, it's very easy to get lost.

K: Would you approach solving, like helping someone solve it differently based on like their experience level?

M: Absolutely. So what I would do is to make sure, what is their level of understanding. How far they are in the math. And then adjust accordingly.

K: That's my next question. So, thank you!

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